Exploring Second Order s-to-z Transformation for IIR Linear Phase Filters Design

Dejan Mirković, Miona Andrejević Stošović and Vančo Litovski

Abstract – This paper explores design of IIR filters with linear phase exploiting a new second order s-to-z domain mapping. Capabilities of the mapping are exemplified with eight order s-domain filter designed to have constant pass-band group delay and all kind of zeros. New mapping will be confronted with bilinear and phase-invariance method. Results of the numerical analysis showed solid behaviour of new mapping over various sampling rates.

Keywords – IIR Filters, Linear Phase, S-to-z mapping.

I. Introduction

With rapid development of integrated circuit (IC) industry, digital signal processing becomes leading concept of signal conditioning in contemporary circuits and systems. Two key stone concepts of the DSP paradigm are: finite impulse response (FIR) and infinite impulse response (IIR). Both concepts are very important and the first step one should take when designing DSP system is to properly decide which one to apply according to given application. Since linear phase is desired characteristic of the system it is natural to embrace FIR concept only since there is no recursion and linear phase is always possible by design. However, this property is usually paid with higher latency and order (more hardware). On the other side there is IIR concept which can provide same functionality with significantly lower order and latency which makes it attractive for low power design. Unfortunately, IIR concept suffers from poor control over phase characteristics. Therefore, additional circuitry is usually introduced (so-called phase correctors) to mitigate this problem [1].

This paper explores possibility of designing linear phase IIR filters utilizing second order transformation function which tries to simultaneously preserve both magnitude and phase characteristics.

Paper is organized as follows. In the second section methods for designing IIR filters will be briefly discussed. Here second order transformation function, and Phase Invariance Method will be presented. Third section will present results of the numeric analysis. Important findings concerning system stability and usefulness of the second order transformation for IIR filter design will be outlined in the conclusion.

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II. Methods for IIR filters design

The number of techniques and methods for designing IIR filters are derived through endeavour of research community along the past decades. When designing the IIR filter one first must design space i.e. domain. Traditionally design of IIR filters starts form *s*-domain, thus two groups of methods can be identified. One group consists of methods with special purpose. The goal of these methods is to preserve, as well as possible, one particular characteristic of the filter at the cost of the others. In this group one can classify Impulse/Step Response Invariant Method [2, 3] and Magnitude/Phase Invariance Method [4, 5]. Another group of methods is based on substitution of complex frequency *s* with corresponding expression in *z*. These methods are usually referred as mapping methods or transforms, e.g. Matched-*z* [6], Forward/Backward Euler and Trapezoidal (Bilinear) transform [7].

Finally, one may choose to do the design directly in *z*-domain. Here designer is usually unlikely to find closed form solutions. Most of the design work is done through recursive numerical routines. One comprehensive set of algorithms and routines supporting this approach can be found in [8].

In this work we focus on the methods which belong to the traditional IIR design approach, i.e. transforming analogue *s*-domain filter prototype into *z*-domain.

A. Second order s-to-z transformation

The second order transfer function is introduced by the authors of [9] for the first time. Authors showed that differentiation function in *s*-domain (i.e. simple multiplication with *s*) can be approximated in closed form in *z*-domain with Eq. (1).

$$s = \frac{1}{2T_s}\frac{3z^2 - 4z + 1}{z^2} \qquad (1)$$

Here, T_s is sampling period. Observing Eq. (1) one may note that dividing numerator with the denominator second order polynomial in terms of z^{-1} (delay elements) emerges. Therefore, there is no rational function like in *bilinear* case given in Eq. (2), hence expected positive impact on the phase characteristic. Of course, this is paid with twice higher order of the resulting filter in the *z*-domain comparing to mapping with *bilinear* transform.

$$s = \frac{2}{T_s}\frac{z-1}{z+1} \quad (2)$$

Since second order, this new transformation function will be further referred to as *quadratic*. Authors of [9] also proved that *quadratic* transformation preserves stability over wide range of sampling rates, equally well as famous *bilinear* transform. It should be noted that *quadratic* transform belongs to the second group of methods at the beginning of the section. In other words it does not favour any characteristics over the others. On contrary, it tries to preserve simultaneously both phase and the magnitude.

It is also important to emphasise that for proper calculation of the frequency response, and group delay real, analogue, angular frequency ω first must be pre-wrapped i.e. mapped in corresponding digital frequency ω_d. This is done by replacing complex frequency s with $j\omega$ and complex variable z with $e^{j\omega_d}$ in Eq. (1) and (2). In a case of *bilinear* transform this relation is already known and derivation easily performed resulting with Eq. (3).

$$\omega_d = 2\tan^{-1}\left(\frac{\omega T_s}{2}\right) \quad (3)$$

Similar is obtained, involving little bit more effort, for *quadratic*. Derivation of this relation is out of the scope of this paper nevertheless it is given in Eq. (4) for completeness.

$$\omega_d = \tan^{-1}\left(\frac{\sqrt{\sqrt{1+2\omega T_s}-1}}{2\sqrt{2}-\sqrt{\sqrt{1+2\omega T_s}+1}}\right) \quad (4)$$

This way each real frequency is properly mapped in z-domain.

B. Phase-Invariance Method

In order to better examine properties of the *quadratic* transform, method from the second group (special purpose methods) will be utilized as well. This way *quadratic* transform will be confronted with most popular representatives from both, first and second group of methods for designing IIR filters. Since linear phase is target parameter to preserve, Phase Invariance Method (PIM) introduced by Paarmann [5] will be exploited.

Phase invariance is based on Hilbert transforms where magnitude response can be obtained from phase response and vice versa [10]. Transformation algorithm is of algorithmic type where samples of the desired, analogue prototype, phase response given in Eq. (5) are taken as the input.

$$\Phi[k] = \Phi(\Omega),\ \omega = \frac{k\pi}{N},\ k = 0,\ldots,N-1 \quad (5)$$

Algorithm may be fine tuned by proper choosing number of samples N. Since inverse Fast Fourier Fourier (FFT) transform is used, N should be in power of two. This method will further be referred to as *Paarmann*.

In this case there is no closed form relation between complex s and z. Variables frequency response is simply calculated in, $\omega_d = \omega T_s$ points.

III. RESULTS OF THE NUMERICAL ANALYSIS

For obtaining results and presenting the data, package for numerical analysis MATLAB is exploited. Package already includes almost any possible standard algorithm and routine tools for filter design in *Signal Processing Toolbox*. These are usually readily available in a form of functions. However, non-standard algorithms such as *quadratic* and *Paarmann*, have to be coded. Therefore, custom function is written implementing *quadratic* transformations, while for *Paarmann* transformation implementation given in [11] is used.

As the test case example is rather complicated, eight order filter with approximately linear pass-band phase response is chosen as an example. Selectivity of the filter is improved with introducing zeros (two pairs of purely imaginary and one pair of right hand side – RHS) like described in [12]. Location of the poles and zeros of the s-domain prototypes is shown in Fig. 1.

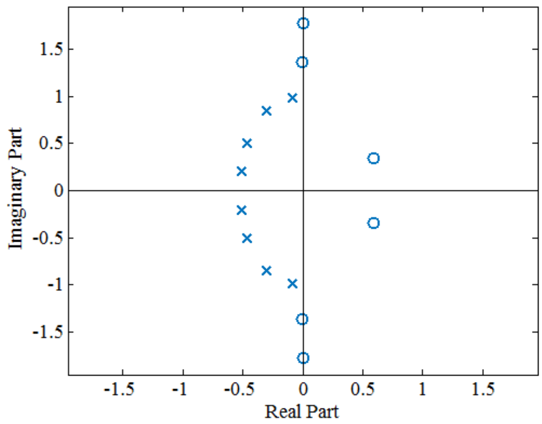
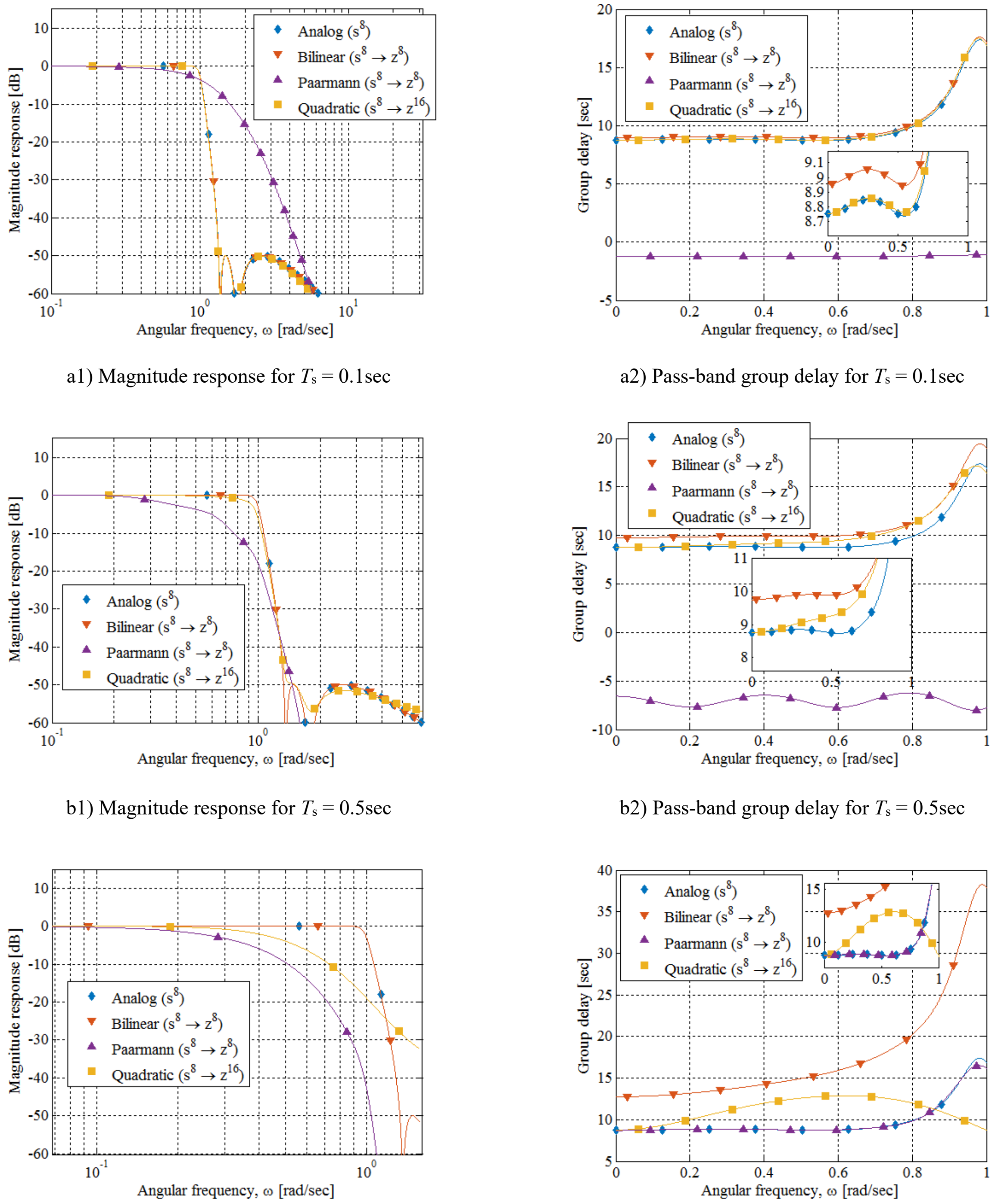


Fig. 1 Poles and zeros location in the s-plane for eight order filter used as test case example.

Fig. 2 shows the results of the numerical analysis for various sampling rates. New, *quadratic*, transformation is compared with *bilinear* and *Paarmann* PIM method. Magnitude response and pass-band group delay are evaluated for three characteristic sampling periods T_s ={0.1, 0.5, 2} sec. Starting, analogue prototype filter characteristics are plotted along with digital counterparts.



a1) Magnitude response for $T_s = 0.1$sec

a2) Pass-band group delay for $T_s = 0.1$sec

b1) Magnitude response for $T_s = 0.5$sec

b2) Pass-band group delay for $T_s = 0.5$sec

c1) Magnitude response for $T_s = 2$sec

c2) Pass-band group delay for $T_s = 2$sec

Fig. 2 Results of the numerical analysis:
a1), b1) and c1) Magnitude response, and a2), b2) and c2) Group delay for T_s= 0.1, 0.5 and 2 sec, respectively.

All plots are shown to $f_s/2$ since in *z*-domain everything is periodic with period of T_s.

First test case is for relatively high sampling rate T_s = 0.1 sec (i.e. f_s = 10Hz). Here, *quadratic* and *bilinear* perform well in both, magnitude and phase response. In this case *Paarmann* fails to follow magnitude since it is designed for approximating phase at the expense of the magnitude. Even though it gives constant group delay it does not approximate analogue prototype group delay in value. This is shown in Fig. 2 a1) and a2). However, if zoomed detail in Fig. 2 a2) is observed slight improvement in group delay over *bilinear* is visible when using *quadratic* mapping. Next, system is sampled with moderate sampling rate (T_s = 0.5 sec). This is depicted in Fig. 2 b1) and b2). In this case *bilinear* still performs well when magnitude response is observed, while *quadratic* starts to deviate from analogue prototype (Fig. 2 b1)). When looking at group delay *quadratic* again outperforms *bilinear* (Fig. 2. b2)).

Finally, when system is sampled with extremely low rate (T_s = 2sec) *Paarmann* PIM method is the best choice when magnitude response is not of primary concern. Interestingly, *quadratic* gives if not constant than at least smaller group delay comparing with *bilinear*.

To check stability of the system obtained using *quadratic* transformation zeros and poles location is shown in Fig. 3 for extreme sampling rate of T_s = 2 sec.

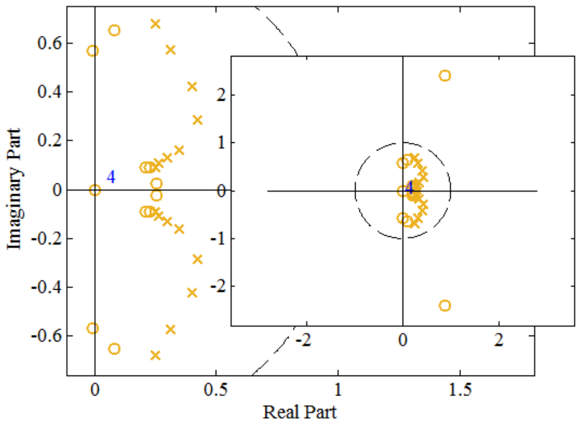


Fig. 3 Zeros and poles location for the *quadratic* transformation for extreme sampling rate T_s = 2sec. Zoomed detail with pole location, behind, and whole picture with two RHS zeros, in-front.

It can be seen that even in this extreme case stability is preserved (all poles inside unit circle). One can note that there are sixteen poles and zeros in the *z*-domain which is direct consequence of second order nature of the transformation.

III. Conclusion

In this paper second order *s*-to-*z* transformation (mapping) is explored for the IIR filter design. First, basic concepts regarding IIR filter design are covered with emphasis on three methods. Two of them are already known to science and engineering community namely, well established *bilinear* and relatively new PIM. The third one, *quadratic*, is introduced by the authors and proved to be usable in the design of IIR filters exhibiting linear phase. Generally, *quadratic* transformation showed a solid behaviour over various sampling rates, while simultaneously trying to preserve both magnitude and phase response of the filter. Robustness of the *quadratic* transformation is confirmed with maintaining filter's stability even under slow sampling rate conditions. Good performance is paid with doubling *z*-domain filter.

Acknowledgement

This work was partially funded by Serbian Ministry of Education Science and Technological Development under contract No. TR32004.

References

[1] Nikolić, S., Stančić, G., "*Design of IIR Notch Filter with Approximately Linear Phase*", Circuits, Systems, and Signal Processing, Vol. 31, No. 6, 2012, pp. 2119-2131.

[2] Hongyan, C. and Lisheng, W., "*Software and hardware implementation of IIR based on Matlab&AccelDSP*", in Weibing, W. (Ed.), Proc. of the 2nd Int. Conf. on Computer Application and System Modelling, Atlantis Press, Paris, 2012. pp. 1411-1415.

[3] Kaiser, J.F., "*Digital filters*", in Kuo, F.F. and Kaiser, J.F. (Eds), System Analysis by Digital Computer, Chapter 7, Willey, New York, pp. 245-252, 1966.

[4] Paarmann, L. D., "*Mapping from the s-domain to the z-domain via magnitude-invariance method*", Signal Processing, Elsevier, Vol. 69 No. 3, 1998, pp. 219-228.

[5] Paarmann, L. D., "*Mapping from the s-domain to the z-domain via phase-invariance method*", Signal Processing, Elsevier, Vol. 86 No. 2, 2006, pp. 223-229.

[6] Shi, R., Wang, S. and Zhao, J., "*An unsplit complex-frequency-shifted PML based on matched Z-transform for FDTD modelling of seismic wave equations*", Journal of Geophysics and Engineering, Vol. 9 No. 2, 2012, pp. 218-229

[7] Park, W., Park, K.-S. and Koh, H.-M., "*Active control of large structures using a bilinear pole-shifting transform with H∞ control method*", Engineering Structures, Vol. 30 No. 11, 2008, pp. 3336-3344.

[8] Micić, A. D., Stojanović, V. S. and Nikolić, S. V. NI, "*On the Direct Design of Recursive Digital Filters*," Faculty of mechanical engineering, 2012 (Niš: Unigraf X-Copy), ISBN 978-86-6055-025-7

[9] Mirković, D., Petković P., and Litovski V., "*A second order s-to-z transform and its implementation to IIR filter design*," COMPEL - The international journal for computation and mathematics in electrical and electronic engineering, vol. 33, no. 5, pp. 1831–1843, 2014. [Online]. Available: http://dx.doi.org/10.1108/COMPEL-03-2014-0058

[10] Oppenheim A.V., Schafer R.W., "*Discrete-Time Signal Processing 2nd Edition*," Prentice-Hall, Upper Saddler River, NJ, 1999.

[11] Vadhavkar, P. R., "*Mapping Controllers from The S-Domain To The Z-Domain Using Magnitude Invariance And Phase Invariance Methods*," MSc Thesis, Wichita State University, 2007.

[12] Litovski, V., "*Synthesis of monotonic passband sharp cutoff filters with constant group delay response*", IEEE Trans. on Circuits and Systems, Vol. 26 No. 8, 1979, pp. 597-602.